
Annealing and the rate distortion problem

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Abstract

In this paper we introduce methodology to determine the bifurcation structure of optima for a class of similar cost functions from Rate Distortion Theory, Deterministic Annealing, Information Distortion and the Information Bottleneck Method. We also introduce a numerical algorithm which uses the explicit form of the bifurcating branches to find optima at a bifurcation point.

1 Introduction

This paper analyzes a class of optimization problems

$$\max_{q \in \Delta} G(q) + \beta D(q) \quad (1)$$

where Δ is a linear constraint space, G and D are continuous, real valued functions of q , smooth in the interior of Δ , and $\max_{q \in \Delta} G(q)$ is known. Furthermore, G and D are invariant under the group of symmetries S_N . The goal is to solve (1) for $\beta = \mathcal{B} \in [0, \infty)$.

This type of problem, which appears to be *NP* hard, arises in Rate Distortion Theory [1, 2], Deterministic Annealing [3], Information Distortion [4, 5, 6] and the Information Bottleneck Method [7, 8].

The following basic algorithm, various forms of which have appeared in [3, 4, 6, 7, 8], can be used to solve (1) for $\beta = \mathcal{B}$.

Algorithm 1 *Let*

$$q_0 \text{ be the maximizer of } \max_{q \in \Delta} G(q) \quad (2)$$

and let $\beta_0 = 0$. For $k \geq 0$, let (q_k, β_k) be a solution to (1). Iterate the following steps until $\beta_K = \mathcal{B}$ for some K .

1. Perform β -step: Let $\beta_{k+1} = \beta_k + d_k$ where $d_k > 0$.

2. Take $q_{k+1}^{(0)} = q_k + \eta$, where η is a small perturbation, as an initial guess for the solution q_{k+1} at β_{k+1} .

3. Optimization: solve

$$\max_{q \in \Delta} G(q) + \beta_{k+1} D(q)$$

to get the maximizer q_{k+1} , using initial guess $q_{k+1}^{(0)}$.

We introduce methodology to efficiently perform algorithm 1. Specifically, we implement numerical continuation techniques [9, 10] to effect steps 1 and 2. We show how to detect bifurcation and we rely on bifurcation theory with symmetries [11, 12, 13] to search for the desired solution branch. This paper concludes with the improved algorithm 6 which solves (1).

2 The cost functions

The four problems we analyze are from Rate Distortion Theory [1, 2], Deterministic Annealing [3], Information Distortion [4, 5, 6] and the Information Bottleneck Method [7, 8]. We discuss the explicit form of the cost function (i.e. $G(q)$ and $D(q)$) for each of these scenarios in this section.

2.1 The distortion function $D(q)$

Rate distortion theory is the information theoretic approach to the study of optimal source coding systems, including systems for quantization and data compression [2]. To define how well a source, the random variable Y , is represented by a particular representation using N symbols, which we call Y_N , one introduces a *distortion function* between Y and Y_N

$$D(q(y_N|y)) = D(Y, Y_N) = E_{y, y_N} d(y, y_N) = \sum_y \sum_{y_N} q(y_N|y) p(y) d(y, y_N)$$

where $d(y, y_N)$ is the *pointwise distortion function* on the individual elements of $y \in Y$ and $y_N \in Y_N$. $q(y_N|y)$ is a stochastic map or *quantization* of Y into a representation Y_N [1, 2]. The constraint space

$$\Delta := \{q(y_N|y) \mid \sum_{y_N} q(y_N|y) = 1 \text{ and } q(y_N|y) \geq 0 \forall y \in Y\} \quad (3)$$

(compare with (1)) is the space of valid quantizers in \mathfrak{R}^n . A representation Y_N is optimal if there is a quantizer $q^*(y_N|y)$ such that $D(q^*) = \min_{q \in \Delta} D(q)$.

In engineering and imaging applications, the distortion function is usually chosen as the *mean squared error* [1, 3, 14], $\hat{D}(Y, Y_N) = E_{y, y_N} \hat{d}(y, y_N)$, where the pointwise distortion function $\hat{d}(y, y_N)$ is the Euclidean squared distance. In this case, $\hat{D}(Y, Y_N)$ is a linear function of the quantizer. In [4, 5, 6], the *information distortion measure*

$$D_I(Y, Y_N) := \sum_{y, y_N} KL(p(x|y_N) || p(x|y)) = I(X; Y) - I(X; Y_N)$$

is used, where the Kullback-Leibler divergence KL is the pointwise distortion function. Unlike the pointwise distortion functions usually investigated in information theory [1, 3], this one is nonlinear, it explicitly considers a third space, X , of inputs, and it depends on the quantizer $q(y_N|y)$ through $p(x|y_N)$ since $p(x|y_N) = \sum_y p(x|y) \frac{q(y_N|y)p(y)}{p(y_N)}$. The only term in D_I which depends on the quantizer is $I(X; Y_N)$, so we can replace D_I with the effective distortion

$$D_{eff}(q) := I(X; Y_N).$$

$D_{eff}(q)$ is the function $D(q)$ from (1) which has been considered in [4, 5, 6, 7, 8].

2.2 Rate Distortion

There are two related methods used to analyze communication systems at a distortion $D(q) \leq D_0$ for some given $D_0 \geq 0$ [1, 2, 3]. In rate distortion theory [1, 2], the problem of finding a minimum rate at a given distortion is posed as a *minimal information rate* distortion problem:

$$\min_{q(y_N|y) \in \Delta} \begin{array}{l} I(Y; Y_N) \\ D(Y; Y_N) \leq D_0 \end{array} . \quad (4)$$

This formulation is justified by the Rate Distortion Theorem [1]. A similar exposition using the Deterministic Annealing approach [3] is a *maximal entropy* problem

$$\max_{q(y_N|y) \in \Delta} \begin{array}{l} H(Y_N|Y) \\ D(Y; Y_N) \leq D_0 \end{array} . \quad (5)$$

The justification for using (5) is Jayne's maximum entropy principle [15]. These formulations are related since $I(Y; Y_N) = H(Y_N) - H(Y_N|Y)$.

Let $I_0 > 0$ be some given information rate. In [4, 6], the neural coding problem is formulated as an entropy problem as in (5)

$$\max_{q(y_N|y) \in \Delta} \begin{array}{l} H(Y_N|Y) \\ D_{eff}(q) \geq I_0 \end{array} . \quad (6)$$

which uses the nonlinear effective information distortion measure D_{eff} .

Tishby et. al. [7, 8] pose an information rate distortion problem as in (4)

$$\min_{q(y_N|y) \in \Delta} \begin{array}{l} I(Y; Y_N) \\ D_{eff}(q) \geq I_0 \end{array} . \quad (7)$$

Using the method of Lagrange multipliers, the rate distortion problems (4),(5),(6),(7) can be reformulated as finding the maxima of

$$\max_{q \in \Delta} F(q, \beta) = \max_{q \in \Delta} [G(q) + \beta D(q)] \quad (8)$$

as in (1) where $\beta = \mathcal{B}$. For the maximal entropy problem (6),

$$F(q, \beta) = H(Y_N|Y) + \beta D_{eff}(q) \quad (9)$$

and so $G(q)$ from (1) is the conditional entropy $H(Y_N|Y)$. For the minimal information rate distortion problem (7),

$$F(q, \beta) = -I(Y; Y_N) + \beta D_{eff}(q) \quad (10)$$

and so $G(q) = -I(Y; Y_N)$.

In [3, 4, 6], one explicitly considers $\mathcal{B} = \infty$. For (9), this involves taking $\lim_{\beta \rightarrow \infty} \max_{q \in \Delta} F(q, \beta) = \max_{q \in \Delta} D_{eff}(q)$ which in turn gives $\min_{q(y_N|y) \in \Delta} D_I$. In Rate Distortion Theory and the Information Bottleneck Method, one could be interested in solutions to (8) for finite \mathcal{B} which takes into account a tradeoff between $I(Y; Y_N)$ and D_{eff} .

For lack of space, here we consider (9) and (10). Our analysis extends easily to similar formulations which use the mean squared error distortion $\hat{D}(q)$, as in [3].

3 Improving the algorithm

We now turn our attention back to algorithm 1 and indicate how numerical continuation [9, 10], and bifurcation theory with symmetries [11, 12, 13] make it more efficient.

We begin by rewriting (8), now incorporating the Lagrange multipliers for the equality constraint $\sum_{y_N} q(y_N|y_k) = 1$ from (3) which must be satisfied for each $y_k \in Y$. This gives the Lagrangian

$$\mathcal{L}(q, \lambda, \beta) = F(q, \beta) + \sum_{k=1}^K \lambda_k \left(\sum_{y_N} q(y_N|y_k) - 1 \right). \quad (11)$$

There are optimization schemes, such as the Fixed Point [4, 6] and projected Augmented Lagrangian [6, 16] methods, which exploit the structure of (11) to find local solutions to (8) for step 3 of algorithm 1.

3.1 Bifurcation structure of solutions

It has been observed that the solutions $\{q_k\}$ undergo *bifurcations* or *phase transitions* [3, 4, 6, 7, 8]. We wish to pose (8) as a dynamical system in order to study the *bifurcation structure* of local solutions for $\beta \in [0, \mathcal{B}]$. To this end, consider the equilibria of the flow

$$\begin{pmatrix} \dot{q} \\ \dot{\lambda} \end{pmatrix} = \nabla_{q, \lambda} \mathcal{L}(q, \lambda, \beta) \quad (12)$$

for $\beta \in [0, \mathcal{B}]$. These are points $\begin{pmatrix} q^* \\ \lambda^* \end{pmatrix}$ where $\nabla_{q, \lambda} \mathcal{L}(q^*, \lambda^*, \beta) = 0$ for some β . The Jacobian of this system is the Hessian $\Delta_{q, \lambda} \mathcal{L}(q, \lambda, \beta)$. Equilibria, (q^*, λ^*) , of (12), for which $\Delta_q F(q^*, \beta)$ is negative definite, are local solutions of (8) [16, 17].

Let $n = NK$. The $(n + K) \times (n + K)$ Hessian of (11) is

$$\Delta_{q, \lambda} \mathcal{L}(q, \lambda, \beta) = \begin{pmatrix} \Delta_q F(q, \beta) & J^T \\ J & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is $K \times K$ [17]. $\Delta_q F$ is the $n \times n$ block diagonal matrix of N $K \times K$ matrices $\{B_i\}_{i=1}^N$ [4]. J is the $K \times n$ Jacobian of the vector of K constraints from (11),

$$J = \underbrace{\begin{pmatrix} I_K & I_K & \dots & I_K \end{pmatrix}}_{N \text{ blocks}}. \quad (13)$$

The kernel of $\Delta_{q, \lambda} \mathcal{L}$ plays a pivotal role in determining the bifurcation structure of solutions to (8). This is due to the fact that bifurcation of an equilibria (q^*, λ^*) of (12) at $\beta = \beta^*$ happen when $\ker \Delta_{q, \lambda} \mathcal{L}(q^*, \lambda^*, \beta^*)$ is nontrivial. Furthermore, the bifurcating branches are tangent to certain linear subspaces of $\ker \Delta_{q, \lambda} \mathcal{L}(q^*, \lambda^*, \beta^*)$ [12].

3.2 Bifurcations with symmetry

Any solution $q^*(y_N|y)$ to (8) gives another equivalent solution simply by permuting the labels of the classes of Y_N . For example, if P_1 and P_2 are two $n \times 1$ vectors such that for a solution $q^*(y_N|y)$, $q^*(y_N = 1|y) = P_1$ and $q^*(y_N = 2|y) = P_2$, then the quantizer where $\hat{q}(y_N = 1|y) = P_2$, $\hat{q}(y_N = 2|y) = P_1$ and $\hat{q}(y_N|y) = q^*(y_N|y)$ for all other classes y_N is a maximizer of (8) with $F(\hat{q}, \beta) = F(q^*, \beta)$. Let S_N be the algebraic group of all permutations on N symbols [18, 19]. We say that $F(q, \beta)$ is S_N -invariant if $F(q, \beta) = F(\sigma(q), \beta)$ where $\sigma(q)$ denotes the action on q by permutation of the classes of Y_N as defined by any $\sigma \in S_N$ [17]. Now suppose that a solution q^* is fixed by all the elements of S_M for $M \leq N$. Bifurcations at $\beta = \beta^*$ in this scenario are called *symmetry breaking* if the bifurcating solutions are fixed (and only fixed) by subgroups of S_M .

To determine where a bifurcation of a solution (q^*, λ^*, β) occurs, one determines β for which $\Delta_q F(q^*, \beta)$ has a nontrivial kernel. This approach is justified by the fact that

$\Delta_{q,\lambda}\mathcal{L}(q^*, \lambda^*, \beta)$ is singular if and only if $\Delta_q F(q^*, \beta)$ is singular [17]. At a bifurcation $(q^*, \lambda^*, \beta^*)$ where q^* is fixed by S_M for $M \leq N$, $\Delta_q F(q^*, \beta^*)$ has M identical blocks. The bifurcation is generic if

$$\begin{aligned} &\text{each of the identical blocks has a single 0-eigenvector, } \mathbf{v}, \\ &\text{and the other blocks are nonsingular.} \end{aligned} \quad (14)$$

Thus, a generic bifurcation can be detected by looking for singularity of one of the $K \times K$ identical blocks of $\Delta_q F(q^*, \beta)$. We call the classes of Y_N which correspond to identical blocks *unresolved* classes. The classes of Y_N that are not unresolved are called *resolved* classes.

The Equivariant Branching Lemma and the Smoller-Wasserman Theorem [12, 13] ascertain the existence of explicit bifurcating solutions in subspaces of $\ker \Delta_{q,\lambda}\mathcal{L}(q^*, \lambda^*, \beta^*)$ which are fixed by special subgroups of S_M [12, 13]. Of particular interest are the bifurcating solutions in subspaces of $\ker \Delta_{q,\lambda}\mathcal{L}(q^*, \lambda^*, \beta^*)$ of dimension 1 guaranteed by the following theorem

Theorem 2 [17] *Let $(q^*, \lambda^*, \beta^*)$ be a generic bifurcation of (12) which is fixed (and only fixed) by S_M , for $1 < M \leq N$. Then there exists M bifurcating solutions,*

$$\begin{pmatrix} q^* \\ \lambda^* \\ \beta^* \end{pmatrix} + \begin{pmatrix} t\mathbf{u}_m \\ \beta(t) \end{pmatrix}, \text{ where } 1 \leq m \leq M, \quad (15)$$

$$[\mathbf{u}_m]_\nu = \begin{cases} (M-1)\mathbf{v} & \text{if } \nu \text{ is the } m^{\text{th}} \text{ unresolved class of } Y_N \\ -\mathbf{v} & \text{if } \nu \text{ is some other unresolved class of } Y_N \\ \mathbf{0} & \text{otherwise} \end{cases} \quad (16)$$

and \mathbf{v} is defined as in (14). Furthermore, each of these solutions is fixed by the symmetry group S_{M-1} .

For a bifurcation from the uniform quantizer, $q_{\frac{1}{N}}$, which is identically $\frac{1}{N}$ for all y and all y_N , all of the classes of Y_N are unresolved. In this case,

$$\mathbf{u}_m = (-\mathbf{v}^T, \dots, -\mathbf{v}^T, (N-1)\mathbf{v}^T, -\mathbf{v}^T, \dots, -\mathbf{v}^T, \mathbf{0}^T)^T$$

where $(N-1)\mathbf{v}$ is in the m^{th} component.

Relevant to the computationalist is that instead of looking for a bifurcation by looking for singularity of the $n \times n$ Hessian $\Delta_q F(q^*, \beta)$, one may look for singularity of one of the $K \times K$ identical blocks, where $K = \frac{n}{N}$. After bifurcation of a local solution to (8) has been detected at $\beta = \beta^*$, knowledge of the bifurcating directions makes finding solutions of interest for $\beta > \beta^*$ much easier (see section 3.4.1).

3.3 The subcritical bifurcation

In all problems under consideration the solution for $\beta = 0$ is known. For (9), (10) this solution is $q_0 = q_{\frac{1}{N}}$. For (4) and (5), q_0 is the mean of Y . Rose [3] was able to compute explicitly the critical value β^* where q_0 loses stability for the Euclidean pointwise distortion function. We have the following related result.

Theorem 3 [20] *Consider problems (9), (10). The solution $q_0 = 1/N$ loses stability at $\beta = \beta^*$ where $1/\beta^*$ is the second largest eigenvalue of a discrete Markov chain on vertices $y \in Y$, where the transition probabilities $p(y_l \rightarrow y_k) := \sum_i p(y_k|x_i)p(x_i|y_l)$.*

Corollary 4 *Bifurcation of the solution $(q_{\frac{1}{N}}, \beta)$ in (9), (10) occurs at $\beta \geq 1$.*

The *discriminant* of the bifurcating branch (15) is defined as [17]

$$\zeta(q^*, \beta^*, \mathbf{u}_m) = \langle \mathbf{u}_m, \partial_{q,\lambda}^3 \mathcal{L}(q^*, \lambda^*, \beta^*)[\mathbf{u}_m, EL^{-1}E\partial_{q,\lambda}^3 \mathcal{L}(q^*, \lambda^*, \beta^*)[\mathbf{u}_m, \mathbf{u}_m]] \rangle - 3\langle \mathbf{u}_m, \partial_{q,\lambda}^4 \mathcal{L}(q^*, \lambda^*, \beta^*)[\mathbf{u}_m, \mathbf{u}_m, \mathbf{u}_m] \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $\partial_{q,\lambda}^n \mathcal{L}[\cdot, \dots, \cdot]$ is the multilinear form of the n^{th} derivative of \mathcal{L} , E is the projection matrix onto $\text{range} \Delta_{q,\lambda} \mathcal{L}(q^*, \lambda^*, \beta^*)$, and L^{-1} is the Moore-Penrose generalized inverse of the Hessian $\Delta_{q,\lambda} \mathcal{L}(q^*, \lambda^*, \beta^*)$.

Theorem 5 [17] *If $\zeta(q^*, \beta^*, \mathbf{u}_m) < 0$, then the bifurcating branch (15) is subcritical (i.e. a first order phase transition). If $\zeta(q^*, \beta^*, \mathbf{u}_m) > 0$, then (15) is supercritical.*

For (9), Theorem 5 predicts a subcritical bifurcation from $(q_{\frac{1}{N}}, \beta^* \approx 1.038706)$ for $N \geq 3$. The existence of a subcritical bifurcation (first order phase transition) is intriguing. In

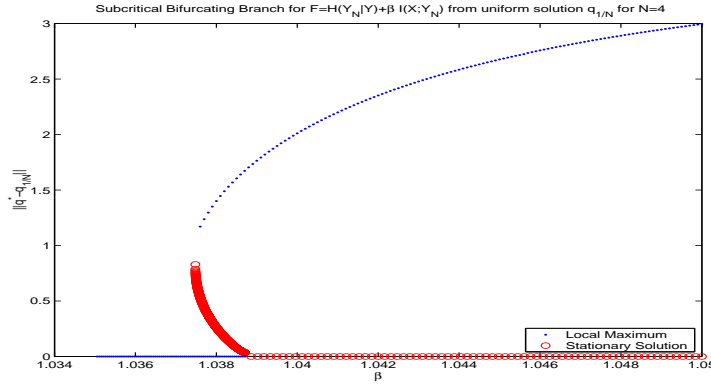


Figure 1: A joint probability space on the random variables (X, Y) was constructed from a mixture of four Gaussians as in [4]. Using this probability space, the equilibria of (12) for F as defined in (9) were found using Newton's method. Depicted is the subcritical bifurcation from $(q_{\frac{1}{4}}, \beta^* \approx 1.038706)$.

analogy to rate distortion curve [2, 1] we can define H - I curve for the problem (6)

$$H(I_0) := \max_{q \in \Delta, D_{eff} \geq I_0} H(Y_N|Y).$$

Let $I_{\max} = \max_{q \in \Delta} D_{eff}$. Then for each $I_0 \in (0, I_{\max})$ the value $H(I_0)$ is well defined and achieved at a point where $D_{eff} = I_0$. At such a point there is a Lagrange multiplier β such that $-\nabla H = \beta \nabla D_{eff}$ and this β solves problem (9). Therefore for each $I \in (0, I_{\max})$ there is a corresponding β which solves problem (9). The existence of subcritical bifurcation in β implies that this correspondence is not monotone for small values of I .

3.4 Numerical Continuation

Numerical *continuation* methods efficiently analyze the solution behavior of dynamical systems such as (12) [9, 10]. Continuation methods can speed up the search for the solution q_{k+1} at β_{k+1} in step 3 of algorithm 1 by improving upon the arbitrary choice $q_{k+1}^{(0)} = q_k + \eta$. First, the vector $(\partial_{\beta} q_k^T \quad \partial_{\beta} \lambda_k^T)^T$ which is tangent to the curve $\nabla_{q,\lambda} \mathcal{L}(q, \lambda, \beta) = \mathbf{0}$ at $(q_k, \lambda_k, \beta_k)$ is computed by solving the matrix system

$$\Delta_{q,\lambda} \mathcal{L}(q_k, \lambda_k, \beta_k) \begin{pmatrix} \partial_{\beta} q_k \\ \partial_{\beta} \lambda_k \end{pmatrix} = -\partial_{\beta} \nabla_{q,\lambda} \mathcal{L}(q_k, \lambda_k, \beta_k). \quad (17)$$

Now the initial guess in step 2 becomes $q_{k+1}^{(0)} = q_k + d_k \cdot \partial_\beta q_k$ where $d_k = \frac{\Delta s}{\sqrt{|\partial_\beta q_k|^2 + |\partial_\beta \lambda_k|^2 + 1}}$ for $\Delta s > 0$. Furthermore, β_{k+1} in step 1 is found by using this same d_k . This choice of d_k assures that a fixed step along $(\partial_\beta q_k^T \ \partial_\beta \lambda_k^T)^T$ is taken for each k . We use three different continuation methods which implement variations of this scheme: *Parameter*, *Tangent* and *Pseudo Arc-Length* [9, 17]. These methods can greatly decrease the optimization iterations needed to find q_{k+1} from $q_{k+1}^{(0)}$ in step 3. The cost savings can be significant, especially when continuation is used in conjunction with a Newton type optimization scheme which explicitly uses the Hessian $\Delta_q F(q_k, \beta_k)$. Otherwise, the CPU time incurred from solving (17) may outweigh this benefit.

3.4.1 Branch switching

Suppose that a bifurcation of a solution q^* of (8) has been detected at β^* . To proceed, one uses the explicit form of the bifurcating directions, $\{\mathbf{u}_m\}_{m=1}^M$ from (16) to search for the bifurcating solution of interest, say q_{k+1} , whose existence is guaranteed by Theorem 2. To do this, let $\mathbf{u} = \mathbf{u}_m$ for some $m \leq M$, then implement a *branch switch* [9]

$$q_{k+1}^{(0)} = q^* + d_k \cdot \mathbf{u}.$$

4 The improved algorithm

We conclude with an efficient algorithm to solve (1). The section numbers in parentheses indicate the location in the text supporting each step.

Algorithm 6 Let q_0 be the maximizer of $\max_{q \in \Delta} G$, $\beta_0 = 1$ (3.3) and $\Delta s > 0$. For $k \geq 0$, let (q_k, β_k) be a solution to (1). Iterate the following steps until $\beta_K = \mathcal{B}$ for some K .

1. (3.4) Perform β -step: solve (17) for $(\partial_\beta q_k^T \ \partial_\beta \lambda_k^T)^T$ and select $\beta_{k+1} = \beta_k + d_k$ where $d_k = \frac{\Delta s}{\sqrt{|\partial_\beta q_k|^2 + |\partial_\beta \lambda_k|^2 + 1}}$.

2. (3.4) The initial guess for q_{k+1} at β_{k+1} is $q_{k+1}^{(0)} = q_k + d_k \cdot \partial_\beta q_k$.

3. Optimization: solve

$$\max_{q \in \Delta} G(q) + \beta_{k+1} D(q)$$

to get the maximizer q_{k+1} , using initial guess $q_{k+1}^{(0)}$.

4. (3.2) Check for bifurcation: compare the sign of the determinant of an identical block of each of

$$\Delta_q [G(q_k) + \beta_k D(q_k)] \text{ and } \Delta_q [G(q_{k+1}) + \beta_{k+1} D(q_{k+1})].$$

If a bifurcation is detected, then set $q_{k+1}^{(0)} = q_k + d_k \cdot \mathbf{u}$ where \mathbf{u} is defined as in (16) for some $m \leq M$, and repeat step 3.

Acknowledgments

Many thanks to Dr. John P. Miller at the Center for Computational Biology at Montana State University-Bozeman. This research is partially supported by NSF grants DGE 9972824, MRI 9871191, and EIA-0129895; and NIH Grant R01 MH57179.

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